

## ON LIMIT AMALGAMATIONS OF STRATIFIED SPACES

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ABSTRACT. In this article we prove that stratified spaces and other geometric subfamilies satisfy categorical Fraïssé properties, a matter that might be of interest for both geometers and logicians. Part of this work was presented by the authors at the First Meeting of Logic and Geometry in Bogotá, on Sept. 2010.

## INTRODUCTION

All topological spaces considered in the sequel will be Hlc2 (Hausdorff, locally compact 2nd countable) spaces. A stratified space is a topological space  $X$  that can be decomposed in a family of disjoint pieces, the *strata*, which are locally closed manifolds satisfying nice incidence properties [15]. Smooth manifolds are locally Euclidean spaces with smooth coordinate changes, and any smooth manifold has a trivial stratification whose singular part is the empty set. In general, each point of a stratified pseudomanifold has a local neighborhood isomorphic to  $\mathbb{R}^n \times c(L)$  where  $c(L)$  is the open cone of another compact stratified space, so an accurate definition is recursive.

We are interested in Ramsey-like properties for stratified spaces and their limits. As a motivation, we show some examples of limits of stratified spaces which are finitely oscillation stable and so, in this approach, our first goal is to establish when a directed family of stratified spaces and embeddings has a limit. We also prove that the family of stratified pseudomanifolds is a Fraïssé category. There is a functor from stratified spaces to countable directed graphs. Basic stratified spaces correspond to finite connected graphs, and every stratified space can be obtained after a countable number of amalgamations and embeddings of basic ones. Under a suitable family of stratified embeddings we show that the family of stratified spaces is a Fraïssé category. The family of stratified pseudomanifolds also has this nice categoric behavior.

## 1. BOUQUETS OF LÉVY FAMILIES. A MOTIVATION

Consider the sequence of unit spheres  $\mathbb{S}^n \cong \mathbb{R}^n \sqcup \{\infty\}$  for  $n \geq 1$  whose singular part is  $\Sigma_n = \{0, \infty\}$ . The linear action of  $U(n)$  on  $\mathbb{S}^n$  leaves  $\Sigma_n$  fixed and is free elsewhere. The obvious equivariant smooth embeddings

$$\mathbb{S}^n \longrightarrow \mathbb{S}^{n+1}$$

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have as limit the unit sphere  $\mathbb{S}^\infty \subset l_2$ , which is a smooth implicit Hilbert manifold [10]. The spheres  $(\mathbb{S}^n, d_n, \mu_n)$ , equipped with the Euclidean metric and the Haar measure, constitute a Lévy family. This fact is one of the main ingredients in Milman's proof of the finite oscillation stability of the limit  $\mathbb{S}^\infty$ , as a  $U(l_2)$ -space [14].

Let  $X_n$  be a sequence of stratified pseudomanifolds and stratified morphism

$X_n \xrightarrow{\iota_n} X_{n+1}$ . In general, we cannot guarantee neither the existence of a decomposition of the limit  $X_\infty = \lim_n X_n$  into disjoint smooth pieces nor the incidence properties of these pieces, even if the morphisms  $\iota_n$  are stratified embeddings. In this article we find a class of strong embeddings, under which the limit space might be either

- A decomposed space whose pieces are disjoint smooth (finite Euclidean or Hilbert) manifolds. If the limit length  $\lim_n \text{len}(X) = \infty$  then the decomposition is not locally finite,
- A Hilbert-smooth-stratified pseudomanifold if the graph of the stratification stabilizes beyond some  $n$ , see §3.7. This is the case of the sphere  $\mathbb{S}^\infty$  above. The regular stratum is  $l_2$ , the link of the singular points  $\{0, \infty\}$  is still  $\mathbb{S}^\infty$ , which is not compact; or
- A stratified pseudomanifold, if  $\lim_n \dim(X_n) < \infty$  is finite.

Take a sequence

$$G_1 \subset \cdots \subset G_n \subset G_{n+1} \subset \cdots$$

of compact Lie groups. Suppose that each  $X_n$  is a  $G_n$ -stratified pseudomanifolds (see [13]) and each morphism  $\iota_n$  is an equivariant strong embedding (see §3.4 below). By integrating with respect to the Haar measure of  $G_n$  we can assume that each  $X_n$  has a metric distance  $d_n$  which is compatible with the smooth stratified structure and  $G_n$  acts by stratified isometries [1]. Choose, for each  $n$ , the probability measure  $\mu_n$  induced by the normalized Haar measures of the group  $G_n$  [3]. According to Theorem 1.2.10 in [14], if the spaces  $(X_n, d_n, \mu_n)$  constitute a Lévy family, then  $X_\infty$  is a finite oscillation stable  $G_\infty$ -space, where  $G_\infty = \lim_n G_n$ . Let us suppose that this is the case.

Assume that for each  $n$ , the space  $X_n$  has at least one fixed point, say  $x_n$ . Fix some integer  $k \geq 2$  and take the so called *bouquet* quotient space

$$Z_n = \frac{X_n \times [k]}{\{x_n\} \times \{1, \dots, k\}}$$

where  $[k] = \{1, \dots, k\}$  and the base point  $z_n = [x_n, j]$  is the class of  $(x_n, j)$ . Take in  $Z_n$  the unique distance  $\hat{d}_n$  that extends  $d_n$  on each copy  $X_n^j = q(X_n \times \{j\})$  where  $q$  is the quotient map. Let  $Z_\infty = \lim_n Z_n$  and consider the action of  $G_\infty^k = G_\infty \oplus \cdots \oplus G_\infty$  on  $Z_\infty$  given by

$$(g_1, \dots, g_k)([z, j]) = [g_j(z), j]$$

Fix some  $\varepsilon > 0$ , an uniformly continuous function  $Z_\infty \xrightarrow{f} \mathbb{R}$  and a finite subset  $F \subset Z_\infty$ . Define  $F^j = (F \cup \{z_\infty\}) \cap X_\infty^j$ . By the finite oscillation stability of  $X_\infty$ ,

for each  $j \in [k]$  there is some  $g_j \in G_\infty$  such that

$$\text{diam} \left( g_j F^j \right) = \sup \left\{ |f(q(x)) - f(q(y))| : x, y \in g_j \left( F^j \right) \right\} < \varepsilon/2$$

We deduce that  $\text{diam}(gF) < \varepsilon$  for  $g = (g_1, \dots, g_k)$ . This shows that  $Z_\infty$  is finite oscillation stable under the stratified action of  $G_\infty^k$ .

Notice that  $\{Z_n\}_n$  is not a Lévy family, since any probability measure in  $Z_n$  comes from a convex combination of the respective measures in the copies  $X_n^j$ ,  $j = 1, \dots, k$  of  $X_n$ .

## 2. FRAÏSSÉ CATEGORIES

Usual Fraïssé limits are formulated in terms of languages and models of finite structures. Given a certain language, a Fraïssé family is, broadly speaking, a set of models and embeddings satisfying nice properties (heritability, joint embeddings, amalgamation and a countable number of non isomorphic models) which guarantees the existence of a limit; see for instance [2, 4]. There is also a categoric treatment of these notions [9]. In order to know more about the connections of Fraïssé theory with Ramsey theory and topological dynamics see [8, 14].

**2.1. Embeddings and sub-objects.** Along this article we will work on a geometric category  $\mathfrak{C}$ , i.e. a non-plain subcategory of  $\mathfrak{Top}$ . The simple idea is that an **embedding** is an arrow in a nice family of morphisms  $\mathcal{E} \supset \text{Iso}(\mathfrak{Top})$ ; which satisfies some categoric properties. Among them, for instance, embeddings are monomorphisms and the composition of embeddings is an embedding. We also say that  $a$  is

a **sub-object** of  $b$  if there is an embedding  $a \xrightarrow{f} b$ . We will not stress on the categoric formalization of these notions, for more details see [5, 6].

**2.2. Fraïssé categories.** A diagram is said to be **in**  $\mathfrak{C}$  if all its objects are in  $\text{Obj}(\mathfrak{C})$  and all its arrows are in  $\text{Hom}(\mathfrak{C})$ . We say that  $\mathfrak{C}$  is a **Fraïssé category** iff all its morphisms are embeddings and it satisfies the following axioms,

- (1) Heritability: Each embedding  $a \longrightarrow b$  with target object  $b \in \text{Obj}(\mathfrak{C})$  is a diagram in  $\mathfrak{C}$ ; so  $a \in \text{Obj}(\mathfrak{C})$ .
- (2) Joint embeddigs: For any  $a, b \in \text{Obj}(\mathfrak{C})$  there is a diagram

$$a \longrightarrow c \longleftarrow b$$

in  $\mathfrak{C}$ . The object  $c \in \text{Obj}(\mathfrak{C})$  is a **joint object**.

- (3) Amalgamation: Each diagram  $c \longleftarrow a \longrightarrow b$  in  $\mathfrak{C}$  can be completed to a commutative square

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

in  $\mathfrak{C}$ . The object  $d \in \text{Obj}(\mathfrak{C})$  is an **amalgamation object**.

2.3. **Examples.** Here there are some examples of Fraïssé categories.

- (1) The category of topological spaces and continuous functions  $\mathfrak{Top}$  is a Fraïssé category, since any two topological spaces can be topologically embedded in their disjoint union; and for any pair of topological embeddings

$$W \xleftarrow{h} X \xrightarrow{f} Y \text{ we can take the amalgamated sum}$$

$$Z = \frac{W \sqcup Y}{\sim} \quad f(x) \sim h(x) \forall x \in X$$

The inclusions  $Y \longrightarrow W \sqcup Y$  and  $W \longrightarrow W \sqcup Y$  induce a commutative square of topological embeddings

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow h & & \downarrow i \\ W & \xrightarrow{j} & Z \end{array}$$

What's more,  $Z$  is a push-out since, for any other  $Z'$  and any pair of con-

tinuous functions  $W \xrightarrow{s} Z' \xleftarrow{t} Y$  inducing a commutative square diagram, i. e. such that  $th = sf$ ; there is a *unique* continuous function

$Z \xrightarrow{\phi} Z'$  such that  $\phi j = s$ ,  $\phi i = t$ ; and if  $s, t$  are topological embeddings then so is  $\phi$ . In the sequel, we will prefer the notation  $Z = W \cup_x Y$  for the amalgamated sum.

- (2) The family of  $CW$ -complexes is closed under disjoint unions, embeddings and topological amalgamations of (suitable)  $CW$ -embeddings. It is therefore a Fraïssé category.
- (3) The family of smooth manifolds is not closed under smooth amalgamations. The 8-curve, which has a singular point, can be obtained as the amalgamation of two disjoint circles  $Y, W$  through a distinguished point  $X = \{p\}$ .

### 3. STRATIFIED SPACES

In order to recover any geometric smooth sense for amalgamations we must allow singular points. One way is to consider them as objects in a larger category, the family of smooth stratified spaces and its morphisms [15].

3.1. **Stratified spaces.** Let  $X$  be a  $\text{Hlc2}$  topological space. A **stratification** of  $X$  is a locally finite partition  $\mathcal{S}$  of  $X$ . The elements of  $\mathcal{S}$  are called **strata**, and they are disjoint locally compact smooth manifolds satisfying an **incidence condition**: Given any two strata  $S, S' \in \mathcal{S}$ , if  $\overline{S} \cap S' \neq \emptyset$  then  $S' \subset \overline{S}$  and we say that  $S'$  **adheres** to  $S$  or just  $S' \leq S$ . If  $\mathcal{S}$  is a stratification of  $X$ , we say that  $(X, \mathcal{S})$  is a **stratified space**, although we might talk about "the stratified space  $X$ " when the choice of  $\mathcal{S}$  is clear in the context. Given a stratified space  $(X, \mathcal{S})$ ,

- (1) The incidence condition is partial order relation.
- (2) For each stratum  $S \in \mathcal{S}$ ;

- (a)  $S$  is maximal (resp. minimal) if and only if it is open (resp. closed).
- (b) The closure of  $S$  is the union of the strata which adhere to it,  $\overline{S} = \bigsqcup_{S' \leq S} S'$ .
- (c) The set  $U_S = \bigsqcup_{S \leq S'} S'$  is open, we call it the **incidence neighborhood** of  $S$ .

Since  $\mathcal{S}$  is locally finite in  $X$ , given a stratum  $S$  the maximal length of all strict order chains in  $\mathcal{S}$  starting at  $S$  is finite. The length of  $S$  is the maximal integer  $l = \text{len}(S)$  such that there is some strict order chain  $S = S_0 < \dots < S_l$  in  $\mathcal{S}$ . The **length** of  $X$  is the supremum of the lengths of the strata, we write it  $\text{len}(X)$ .

A stratum  $S$  is maximal (resp. minimal) iff it is open (resp. closed). Open strata will be called **regular** and, by opposition, a **singular** stratum will be a non open one. The **singular part**  $\Sigma$  (resp. **minimal part**  $\mathcal{M}$ ) is the union of all singular (resp. minimal) strata.

A **morphism**  $X \xrightarrow{f} Y$  between stratified spaces  $(X, \mathcal{S}_X)$  and  $(Y, \mathcal{S}_Y)$  is a continuous function that sends smoothly each stratum of  $X$  to some stratum of  $Y$ .

The induced arrow  $(\mathcal{S}_X, \leq) \xrightarrow{f^*} (\mathcal{S}_Y, \leq)$  is a poset morphism. An **isomorphism** is a bijective morphism whose inverse is also a morphism, the  $f^*$  induced by an isomorphism  $f$  is a poset isomorphism.

### 3.2. Examples of stratified spaces.

- (1) Each manifold  $M$  is a stratified space with respect to the family  $\mathcal{S}_M = \{M\}$ , we call it the trivial stratification of  $M$ .
- (2) The Cartesian product  $X \times Y$  of stratified spaces  $(X, \mathcal{S}_X)$  and  $(Y, \mathcal{S}_Y)$  is a stratified space, with the canonical stratification

$$\mathcal{S}_{X \times Y} = \{S \times T : S \in \mathcal{S}_X, T \in \mathcal{S}_Y\}$$

- (3) The disjoint union  $X \sqcup Y$  of stratified spaces  $(X, \mathcal{S}_X)$  and  $(Y, \mathcal{S}_Y)$  is a stratified space, with the canonical stratification

$$\mathcal{S}_{X \sqcup Y} = \{S : S \in \mathcal{S}_X \text{ or } S \in \mathcal{S}_Y\}$$

- (4) Any open subset of a stratified space is also a stratified space.
- (5) Given a compact stratified space  $(L, \mathcal{S}_L)$ , the **open cone**  $c(L) = \frac{L \times [0, \infty)}{L \times \{0\}}$  has a canonical stratification

$$\mathcal{S}_{c(L)} = \{v\} \cup \{S \times \mathbb{R}^+ : S \in \mathcal{S}_L\}$$

We write  $[p, r]$  for the equivalence class of a point  $(p, r)$ , and  $v$  for the equivalence class of  $L \times \{0\}$ , which we call the **vertex** of the cone. We also adopt the convention  $c(\emptyset) = \{\star\}$ .

**3.3. Stratified subspaces.** Let  $(X, \mathcal{S})$  be a stratified space and  $Z \subset X$  any topological subspace. The family

$$\mathcal{S}_Z = \{S \cap Z : S \in \mathcal{S}\}$$

is the **induced partition** of  $Z$ . We will also consider the family

$$\mathcal{S}_{(X, Z)} = \{S \cap Z, S \cap (\partial Z), (S - \overline{Z}) : S \in \mathcal{S}\}$$

where  $\partial Z = \overline{Z} \cap \overline{X - Z}$  is the topological boundary. This family is the **refinement** of  $\mathcal{S}_X$  induced by  $Z$ . We will say that  $Z$  is a **stratified subspace** (resp. a **regular stratified subspace**) of  $X$  iff  $\mathcal{S}_Z$  is a stratification of  $Z$  (resp. iff  $\mathcal{S}_{(X,Z)}$  is a stratification of  $X$ ), which happens iff the intersection of  $Z$  (resp. and  $\partial Z$ ) with any stratum  $S \in \mathcal{S}$  is a submanifold of  $S$ .

**3.4. Embeddings.** Let  $X \xrightarrow{f} Y$  be a morphism; we say that...

- (1)  $f$  is an **immersion** iff the restriction  $f|_S$  to each stratum  $S \in \mathcal{S}_X$  is a smooth immersion.
- (2)  $f$  is a **weak embedding** iff it is a 1-1 immersion satisfying the **lifting property**: For each morphism  $X' \xrightarrow{h} Y$  such that  $\text{Im}(h) \subset \text{Im}(f)$ , the composition  $\hat{f} = fh^{-1}$  is a morphism  $X' \xrightarrow{\hat{f}} X$ .
- (3)  $f$  is an **embedding** iff  $f(X)$  is a stratified subspace of  $Y$  and  $X \xrightarrow{f} f(X)$  is an isomorphism.
- (4)  $f$  is a **strong embedding** iff  $f$  is an embedding,  $f(X)$  is regular and  $\mathcal{S}_Y = \mathcal{S}_{(Y, f(X))}$ .

**3.5. Examples.**

- (1) Each embedding is a weak embedding.
- (2) If  $X \xrightarrow{f} Y$  is a weak embedding, then the stratification of  $X$  is uniquely determined by  $Y$  up to isomorphisms.
- (3)  $Z \subset X$  is a stratified subspace (resp. regular) iff the inclusion  $Z \longrightarrow X$  is an embedding (resp. a strong embedding).
- (4) Consider the 8-curve  $\gamma \subset \mathbb{R}^2$  and let  $p \in \gamma$  be the singular point. Let  $C_1, C_2$  be the two connected components of  $\gamma - p$ . Consider in  $\gamma$  the stratifications  $\mathcal{S}_0 = \{p, (\gamma - p)\}$  and  $\mathcal{S}_1 = \{p, C_1, C_2\}$ . Also consider in  $\mathbb{R}^2$  the stratifications  $\mathcal{T}_0 = \{\mathbb{R}^2\}$ ,  $\mathcal{T}_1 = \{p, (\mathbb{R}^2 - p)\}$ ,  $\mathcal{T}_2 = \{p, (\gamma - p), (\mathbb{R}^2 - \gamma)\}$  and  $\mathcal{T}_3 = \{p, C_1, C_2, (\mathbb{R}^2 - \gamma)\}$ . Write  $\gamma_i$  and  $\mathbb{R}_j^2$  for the stratified spaces  $(\gamma, \mathcal{S}_i)$ , and  $(\mathbb{R}^2, \mathcal{T}_j)$ . Then
  - The identity  $\gamma_k = \gamma_l$  is not a morphism for  $k = 0, l = 1$ ; an embedding for  $k = 1, l = 0$  and  $k = 0$ ; and an isomorphism for  $k = l$ .
  - The inclusion  $\gamma_k \subset \mathbb{R}_l^2$  is a proper (1-1)-immersion for  $k \leq l$ . It is an embedding if  $k = l + 1$ , and a strong embedding if  $k = l + 2$ . For  $k = 0$  and  $l = 3$  it is not a morphism.

A more precise explanation of this situation is given below.

**3.6. Reducibility.** The disjoint union of two stratified spaces is again a stratified space, see §3.2-(3). A stratified space  $(X, \mathcal{S})$  is **reducible** if it can be written as the disjoint union of two stratified spaces; and **irreducible** if not. If  $X = X_1 \sqcup X_2$  is such a disjoint union then, by §3.3  $X_1, X_2$  are regular stratified subspaces of  $X$ .

An **irreducible component** of  $X$  is a minimal irreducible (regular) stratified subspace of  $X$ . If the strata of  $X$  are all connected manifolds, then an irreducible

component is a connected component of  $X$ ; although in general irreducible components might not be connected.

**3.7. Associated graphs.** Given a stratified space  $(X, \mathcal{S})$ ; the **graph**  $\Gamma_X$  associated to  $X$  is the directed graph induced by the poset  $(\mathcal{S}, \leq)$ .

A **vertex** of  $\Gamma_X$  is a stratum  $S \in \mathcal{S}$ . A **directed edge**  $\{S_2, S_1\}$  is a minimal strict incidence chain  $S_1 < S_2$ , which means that there are no intermediate strata between  $S_1, S_2$ .

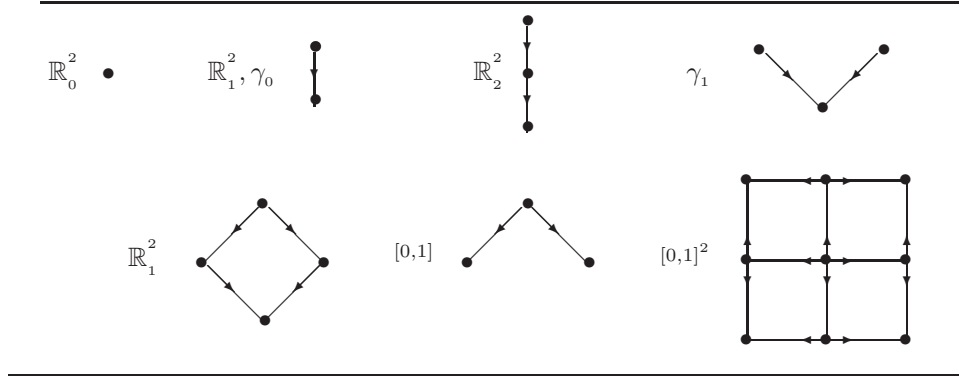


Figure 1: Some examples of graphs induced by stratifications, cf. §3.5-(4)

**3.8. Properties of the associated graph.** Given a stratified space  $(X, \mathcal{S})$ .

- (1)  $\Gamma_X$  has (at most) a countable number of vertices.
- (2) Each path in  $\Gamma_X$  is finite.
- (3)  $\text{len}(X)$  is the (supremum) of the lengths of the paths in  $\Gamma_X$ .
- (4) For each stratum  $S \in \mathcal{S}$ 
  - (a)  $\Gamma_{\bar{S}}$  is the subgraph in  $\Gamma_X$  consisting of all paths starting at  $S$ .
  - (b)  $\Gamma_{U_S}$  is the finite subgraph in  $\Gamma_X$  consisting of all paths ending at  $S$ .
- (5)  $X$  is irreducible if and only if  $\Gamma_X$  is a connected graph.
- (6) The graph of a normal stratified pseudomanifold is a tree <sup>(1)</sup>.
- (7) A regular subspace  $Z \subset X$  is closed if and only if  $\Gamma_Z$  is a subgraph of  $\Gamma_X$ . Strong embeddings of stratified spaces correspond to strong embeddings of graphs in the sense of [7, 3.4-p.414], so the family has a partial order that takes into account the embeddings of regular subspaces [4, p.364].

These properties can be easily checked, we leave the details to the reader.

**3.9. Amalgamations of stratified spaces.** Stratified spaces and strong embeddings provide a full geometric sense of smoothness for amalgamations.

**Lemma 3.9.1.** *Let  $X \xrightarrow{f} Y$  be a proper 1-1 immersion. Then  $f$  is an embedding iff the induced poset morphism  $(\mathcal{S}_X, \leq) \xrightarrow{f^*} (\mathcal{S}_Y, \leq)$  is 1-1.*

[Proof] Notice that

<sup>1</sup>Stratified normality is the property that all the links of  $X$  are connected; it should not be confused with the  $T_4$  separation axiom, which is satisfied by any stratified space. See the next sections and [11] for more details.

- (1)  $X \xrightarrow{f} f(X)$  is an homeomorphism: This is a consequence of two facts,
  - (a)  $X$  is locally compact and  $Y$  is Hausdorff.
  - (b) A proper continuous bijection from a compact space to a Hausdorff spaces is a homeomorphism.
- (2) For each stratum  $S \in \mathcal{S}_X$  the restriction  $S \xrightarrow{f} f(S)$  is a diffeomorphism:  
 Let  $R \in \mathcal{S}_Y$  be the corresponding stratum such that  $f(S) \subset R$ . Since  $f|_S$  is a smooth proper 1-1 immersion, we deduce that  $S \xrightarrow{f} R$  is a smooth embedding, so  $f(S)$  is a regular submanifold of  $R$ . Now  $S \xrightarrow{f} f(S)$  is a proper embedding between equidimensional manifolds, so it is a diffeomorphism.
- (3)  $f(X)$  is a stratified subspace of  $Y$ : This is a consequence of the previous step. The family  $\mathcal{S}_{f(X)}$  is a stratification of  $f(X)$ .
- (4)  $X \xrightarrow{f} f(X)$  is an isomorphism iff  $f^*$  is 1-1: Write  $h = f^{-1}$ . Since  $f$  is a homeomorphism on its image, the inverse map  $f(X) \xrightarrow{h} X$  is continuous. The map  $h$  sends each stratum of  $f(X)$  in a disjoint union of non comparable strata in  $X$ . What's more, by step (2), the restriction of  $h$  to each stratum of  $f(X)$  is smooth. Finally,  $h$  is stratum preserving iff each stratum of  $Y$  meets  $f(X)$  in only one stratum, i. e. iff the induced map  $(f(X), \mathcal{S}_{f(X)}) \xrightarrow{h} (X, \mathcal{S}_X)$  is well defined, which happens iff  $f^*$  is 1-1.

□

As a consequence,

**Theorem 3.9.2.**

- (1) *The amalgamation of two stratified spaces by a pair of strong embeddings is a stratified space.*
- (2) *Stratified spaces constitute a Fraïssé category with respect to strong embeddings.*

[Proof] Statement (2) is a direct consequence of (1), which we now prove.

Any two stratified spaces  $(W, \mathcal{S}_W)$  and  $(Y, \mathcal{S}_Y)$  can be strongly embedded in their disjoint union, so we have joint embeddings, see axiom §2.2-(1). We now verify §2.2-(2), the amalgamation property. Let  $(X, \mathcal{S}_X)$  be any other stratified space and assume that the arrows  $f, h$  in §2.3-(1) are strong embeddings. Then

- The amalgamated sum  $W \sqcup_X Y$  is a stratified space: Let  $W \sqcup Y \xrightarrow{q} Z$  be the quotient map. The family

$$\mathcal{S}_{W \sqcup_X Y} = \{q(S) : S \in \mathcal{S}_W \text{ or } S \in \mathcal{S}_Y\}$$



is a locally finite partition. Since  $f, h$  are strong embeddings,  $X$  can be seen as a regular stratified subspace of  $W$  and  $Y$  at the same time, and therefore the elements of  $\mathcal{S}_{W \sqcup_X Y}$  are locally closed manifolds (with the induced topology) and satisfy the incidence condition §3.1.

- The quotient map  $W \sqcup_X Y \xrightarrow{q} Z$  is a morphism: This is straightforward.
- §2.3-(1) is a diagram of stratified strong embeddings: Since the inclusions of  $W, Y$  in the disjoint union  $W \sqcup Y$  are strong embeddings, the obvious induced maps

$$W \xrightarrow{j} W \sqcup_X Y \xleftarrow{i} Y$$

are strong embeddings.

This concludes the proof.  $\square$

**Corollary 3.9.3.** *The amalgamated sum of stratified spaces with strong embeddings is a pushout.*

This is an easy consequence, we leave the details to the reader.

**3.10. Basic stratified spaces.** A stratified space  $(X, \mathcal{S})$  is **basic** iff

- (1) It is irreducible.
- (2) It has finite length.

Therefore,  $X$  is basic iff  $\Gamma_X$  is a connected finite graph.

**Lemma 3.10.1.** *Let  $W \sqcup_X Y$  be the amalgamation of two stratified spaces along a closed regular subspace  $X$ .*

- (1) *If  $Y, W$  are irreducible, then so is  $W \sqcup_X Y$ .*
- (2) *If  $Y, W$  are basic, then so is  $W \sqcup_X Y$ .*

[Proof] Let  $W \xleftarrow{f} X \xrightarrow{h} Y$  be two strong embeddings, and assume that  $X$  is a closed subspace of (both)  $W, Y$ . Then the graph  $\Gamma_{W \sqcup_X Y}$  of the amalgamated space is the joint of  $\Gamma_W$  and  $\Gamma_Y$  along  $\Gamma_X$ ; i. e.

$$\Gamma_{W \sqcup_X Y} = \Gamma_W \vee_{\Gamma_X} \Gamma_Y$$

We conclude that if  $\Gamma_W, \Gamma_Y$  are connected (resp. finite) graphs then so is  $\Gamma_{W \sqcup_X Y}$ .  $\square$

Here there is another useful and easy result.

**Lemma 3.10.2.** *Non comparable strata can be separated with disjoint open subsets.*

[Proof] Notice that

- (a) It is enough to show it for minimal strata: If  $\mathcal{F} \subset \mathcal{S}$  is a family of non-comparable strata, take the union of the incidence neighborhoods  $Z = \bigcup_{S \in \mathcal{F}} U_S$ . Then  $Z$  is open in  $X$ , and  $S \in \mathcal{F}$  iff  $S$  is a minimal stratum in  $Z$ . Since  $Z$  is open, it is enough to give a family of disjoint neighborhoods in  $Z$  separating the strata in  $\mathcal{F}$ .

- (b) Minimal strata can be separated by disjoint open subsets: Any two different minimal strata in  $X$  are disjoint closed subsets, that can be separated with two disjoint open subsets because  $X$  is  $T_4$ . The whole family of minimal strata can be separated because of §3.1-(1), (2)-(b), and the facts that  $X$  is  $T_4$ , and  $\mathcal{S}$  is locally finite [12].

□

**Proposition 3.10.3.** *Each stratified space is the result of, at most, a countable number of disjoint unions or amalgamations of basic ones.*

[Proof] Let  $(X, \mathcal{S})$  be a stratified space. Take a minimal stratum  $S$  in  $X$ . By §3.8-(4b) and §3.10;  $U_S$  is basic. If  $S' \neq S$  is another minimal stratum and  $U_S \cap U_{S'} \neq \emptyset$  is non-empty; then the graph  $\Gamma_{U_S \cup U_{S'}}$  is connected and  $U_S \cup U_{S'}$  is irreducible. (2) Since  $\Gamma_{U_S}, \Gamma_{U_{S'}}$  are finite then so is  $\Gamma_{U_S \cup U_{S'}}$ . By §3.10 we deduce that

$$U_S \cup U_{S'} \cong U_S \bigcup_{U_S \cap U_{S'}} U_{S'}$$

is basic. Since the stratification is locally finite, by §3.10.2 and §3.1-(2) there is at most a countable number of minimal strata. Since

$$X = \cup \{U_S : S \text{ is minimal} \}$$

we conclude that  $X$  is the result of, at most, a countable number of unions or amalgamations of basic stratified spaces. □

#### 4. STRATIFIED PSEUDOMANIFOLDS

A stratified pseudomanifold is a stratified space together a family of conic charts which reflect the way in which we approach the singular part. The definition is given by induction on the length.

**4.1. Stratified pseudomanifolds.** A 0-length **stratified pseudomanifold** is a smooth manifold with the trivial stratification. A stratified space  $(X, \mathcal{S})$  with  $l(X) > 0$  is a **stratified pseudomanifold** if, for each singular stratum  $S$ ,

- (1) There is a compact stratified pseudomanifold  $(L, \mathcal{S}_L)$  with  $l(L) < l(X)$ . We call  $L$  the **link** of  $S$  because
- (2) Each point  $x \in S$  has an open neighborhood  $x \in U \subset S$  and a stratified

embedding  $U \times c(L) \xrightarrow{\alpha} X$  on an open neighborhood of  $x \in X$ .

The image of  $\alpha$  is called a **basic neighborhood** of  $x$ . Notice that  $\Im(\alpha) \cap S = U$ . Without loss of generality, we assume that  $\alpha(u, v) = u$  for each  $u \in U$  (where  $v$  is the vertex of  $c(L)$ , see §3.2-(5)). We summarize the above situation by saying that the pair  $(U, \alpha)$  is a **chart** of  $x$ . The family of charts is an **atlas** of  $(X, \mathcal{S})$ .

#### 4.2. Examples.

- (1) A basic model  $U \times c(L)$  is a stratified pseudomanifold if  $(L, \mathcal{S}_L)$  is a compact stratified pseudomanifold.

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<sup>2</sup>Notice that for any stratum  $T$  such that  $T \subset U_S \cup U_{S'}$ , we have  $U_T \subset U_S \cup U_{S'}$ , so the family  $\{U_S : S \in \mathcal{S}\}$  is a basis.

- (2) If  $M$  is a manifold and  $X$  is a stratified pseudomanifold then  $M \times X$  is a stratified pseudomanifold.
- (3) Every open subset of a stratified pseudomanifold is again a stratified pseudomanifold.
- (4) Since algebraic manifolds satisfy the Whitney's conditions, every algebraic manifold is a stratified pseudomanifold [15].

Now we will extend some results of the previous section.

**Proposition 4.2.1.** *Let  $W, X, Y$  be stratified pseudomanifolds with finite length. If*

- (1)  $W \xleftarrow{f} X \xrightarrow{h} Y$  are strong embeddings, and
- (2)  $X$  is closed in both  $W, Y$ ;

*then  $W \cup_x Y$  is a stratified pseudomanifold.*

[Proof] The amalgamated space  $Z = W \cup_x Y$  is a finite length stratified space. We check the existence of conical charts §4.1 for each  $z \in Z$ . Since  $X$  is closed in  $Y$ ; if  $z \in (Z - W) \cong (Y - X)$ , then by example §4.2-(3) there is nothing to prove. The same holds for  $z \in (Z - Y)$  so we must check it only for  $z \in W \cap Y = X$ .

Let us remark that this is a local problem. Since  $z$  has conical charts in  $W, X, Y$ , and up to some minor details we assume that  $W = S \times c(L)$ ,  $Y = S \times c(L')$  and  $X = S \times c(N)$  are basic neighborhoods and  $f(u, z) = h(u, z) = u$  for any  $u \in S$ .

We now proceed by induction on  $l = \text{len}(X)$ .

- Case  $l = 0$ : Then  $N = \emptyset$  and  $X = S$ . The amalgamated sum

$$W \cup_x Y = [S \times c(L)] \cup [S \times c(L')] = S \times c(L \sqcup N)$$

is a basic neighborhood of  $z \in S$ , and the link of  $z$  is the disjoint union of the links in  $W, Y$ .

- Inductive hypothesis: We assume §4.2.1 for any triple  $(W', X', Y')$  such that  $\text{len}(X') \leq l - 1$ .
- General case  $l > 0$ : Notice that, by the hypotheses of §4.2.1, the (image of the) pseudomanifold  $N$  is closed in  $L, L'$ . Since  $\text{len}(N) < \text{len}(X)$ ; we can apply the inductive hypothesis to  $(L, N, L')$  in order to get a stratified pseudomanifold  $L \cup_N L'$ . Therefore

$$W \cup_x Y = S \times c \left( L \cup_N L' \right)$$

is a stratified pseudomanifold.

□

*Remark 4.2.2.* The condition that  $X$  is closed, in §4.2.1-(2), cannot be avoided. As a simple counter-example, consider  $W = [0, 1] \times [0, 1]$  a closed square,  $Y = [1, 2] \times (0, 1)$  and  $X = \{1\} \times (0, 1)$ . There is a link at  $p = (1, 0) \in W \cup_x Y$  but it is not compact.

**Proposition 4.2.3.** *Each stratified pseudomanifold is the result of, at most, a countable number of disjoint unions or amalgamations of basic pseudomanifolds.*

[*Proof*] By §3.10.3  $X$  is the (at most countable) union of the incidence neighborhoods  $U_s$  of all its (minimal) strata. By §4.2 these open neighborhoods are basic pseudomanifolds.  $\square$

We summarize the results of this section in the following

**Theorem 4.2.4.** *The family of stratified pseudomanifolds, with respect to strong embeddings, is a Fraïssé category. Moreover, it is the closure by amalgamations of the family of basic stratified pseudomanifolds.*

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